THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2016-2017
Suggested Solution to Assignment 8

1 (a) Let $f(z) = -6z^4$ and $g(z) = z^6 + 2z^3 - z$. Note that for $|z| = 1$,

$$|f(z)| = |-6z^4| = 6|z|^4 = 6 \text{ and } |g(z)| \leq |z|^6 + 2|z|^3 + |z| = 4 < |f(z)|$$

Therefore, by Rouche's theorem, the number of zeros of f and $f + g$ inside $|z| = 1$ are the same. Since 0 is a zero of order 4 of $f(z)$ inside $|z| = 1$, the number of zeros of $(f + g)(z) = z^6 - 6z^4 + 2z^3 - z$ inside $|z| = 1$ is 4.

(b) Let $f(z) = z^5$ and $g(z) = -3z^3 - z + 1$. Note that for $|z| = 2$,

$$|f(z)| = |z^5| = 32 \text{ and } |g(z)| \leq 3|z|^3 + |z| + 1 = 27 < |f(z)|$$

Therefore, by Rouche's theorem, the number of zeros of f and $f + g$ inside $|z| = 2$ are the same. Since 0 is a zero of order 5 of $f(z)$ inside $|z| = 2$, the number of zeros of $(f + g)(z) = z^5 - 3z^3 - z + 1$ inside $|z| = 1$ is 5.

2 First of all, for any $n \in \mathbb{N}$, we consider the function $f_n(z)$ defined by $f_n(z) = z - 1 - \frac{1}{n}$. Let $g(z) = e^{-z}$. Consider the positively oriented contour

$$C = \{Re^{i\theta} \mid \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\} \cup \{iR(-t) \mid t \in [-1, 1]\}$$

For any $R > 4$, along the contour C, we have

$$|f(z)| = |z - 1 - \frac{1}{n}| \geq 1 + \frac{1}{n} > 1 \text{ and } |g(z)| = e^{-x} \leq e^0 = 1$$

As a result, by Rouche's theorem, the number of zeros of f_n and $f_n + g$ inside C are the same. Since $1 + \frac{1}{n}$ is the only zero of $f(z)$ and its multiplicity is 1, the number of zeros of the function $(f_n + g)(z) = z - 1 - \frac{1}{n} + e^{-z}$ inside C is 1.

Now we consider the function $f(z) = z - 1$. Note that for any $z \in \mathbb{C}$,

$$|(f(z) + g(z)) - (f_n(z) + g(z))| = \frac{1}{n}$$

Therefore, the functions $\{(f_n + g)(z)\}_{n \in \mathbb{N}}$ converge uniformly to the function $(f + g)(z)$. As a result, by Hurwitz's theorem, for any $R > 4$, there exists $N \in \mathbb{N}$ such that $(f_n + g)(z)$ and $(f + g)(z)$ have the same number of zeros inside C. This implies that $(f + g)(z) = z - 1 + e^{-z}$ has exactly one root in the right half plane.

Remark: Since this question is quite tricky, you will not lose any mark even if your answer is incorrect.

3 Let $f(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$ and $g(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$ be two linear fractional transformations. By direct computation, one can show that $f(g(z)) = \dfrac{a_3 z + b_3}{c_3 z + d_3}$, where

$$\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \text{ and}$$

$$\det \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \det \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \det \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \neq 0.$$

This shows that composition of two linear fractional transformations is a linear fractional transformation.

4 Note that the equation of straight line and circle can be written in the form

$$Az\overline{z} + \overline{B}z + B\overline{z} + C = 0,$$

where $A, C \in \mathbb{R}$, $B \in \mathbb{C}$ and $AC < |B|^2$. Under the transformation $\omega = \dfrac{1}{z}$, we can see that the equation becomes

$$A\frac{1}{\omega}\frac{1}{\omega} + \overline{B}\frac{1}{\omega} + B\frac{1}{\overline{\omega}} + C = 0,$$

which is equivalent to

$$C\omega\overline{\omega} + B\omega + \overline{B}\overline{\omega} + A = 0.$$

Therefore, the transformation $\omega = \dfrac{1}{z}$ maps straight line and circle to straight line and circle.

5 Let $F(z) = (z, f(z_1), f(z_2), f(z_3)$. Note that since $f(z)$ and $F(z)$ are linear fractional transformations, the mapping $F(f(z)) = (f(z), f(z_1), f(z_2), f(z_3))$ is a linear fractional transformation. Furthermore, $F(f(z)) = \dfrac{f(z) - f(z_1)}{f(z) - f(z_3)} \dfrac{f(z_2) - f(z_3)}{f(z_2) - f(z_1)}$ maps z_1, z_2, z_3 to $0, 1, \infty$. Since there exists a unique linear transformation which maps z_1, z_2, z_3 to $0, 1, \infty$, we have $(z, z_1, z_2, z_3) = (f(z), f(z_1), f(z_2), f(z_3))$.